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Local Precision and Global Harmony: A Comparative Literature Review (LR) Framework for Taylor and Fourier Series in Engineering Modeling

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Abstract

Mathematical series are foundational tools in both theoretical and applied sciences, yet a coherent, comparative understanding of their distinct behaviors especially in engineering contexts remains underexplored. This paper addresses a critical research gap by systematically investigating how and why different series types (arithmetic, geometric, power, Taylor, and Fourier) exhibit fundamentally divergent convergence properties, representational capabilities, and domain-specific efficacies when modeling real-world phenomena. The central research question is: Under what conditions should a given series type be preferred for engineering analysis, particularly when dealing with periodicity, discontinuities, or local versus global behavior?. The novelty of this work lies in its integrative framework that contrasts local approximations (Taylor/power series) with global, periodic representations (Fourier series), explicitly linking mathematical theory to engineering practice. This research study demonstrates that while Taylor series excel in high-precision local modeling near analytic points, they fail to represent discontinuous or periodic systems common in mechanical vibrations, signal processing, and thermal dynamics. In contrast, Fourier series robustly capture such behaviors through harmonic decomposition, despite exhibiting the Gibbs phenomenon near discontinuities. This research contributes directly to the engineering domain by providing a decision-oriented taxonomy that guides practitioners in selecting the optimal series expansion for problems in vibration analysis, heat transfer, fault diagnostics, acoustics, and control systems. By clarifying convergence limitations, operational rules (e.g., term-wise differentiation and integration), and the physical interpretability of harmonic components, this review enhances both pedagogical clarity and computational reliability in engineering modeling. Ultimately, the work bridges abstract mathematical theory and applied engineering design, empowering more informed, effective use of series-based methods in modern technological challenges.

Keywords: Arithmetic series, geometric series, power series, Fourier series, Taylor series, convergence, periodic functions, series expansion, mathematical analysis, signal processing.

الدقة المحلية والتناغم العالمي: إطار مراجعة الأدبيات المقارنة لمتسلسلتي تايلور وفورييه في النمذجة الهندسية

ملخص

تُعدّ المتسلسلات الرياضية أدوات أساسية في العلوم النظرية والتطبيقية، إلا أن الفهم المقارن والمتماسك لسلوكياتها المتميزة، لا سيما في السياقات الهندسية، لا يزال غير مستكشف. تتناول هذه الورقة البحثية فجوة بحثية حرجة من خلال البحث المنهجي في كيفية وأسباب إظهار أنواع مختلفة من المتسلسلات (الحسابية، والهندسية، ومتسلسلات القوى، ومتسلسلات تايلور، ومتسلسلات فورييه) خصائص تقارب متباينة جوهريًا، وقدرات تمثيلية، وفعاليات خاصة بالمجال عند نمذجة ظواهر العالم الحقيقي. السؤال البحثي المحوري هو: في أي ظروف يُفضّل نوع معين من المتسلسلات للتحليل الهندسي، وخاصةً عند التعامل مع الدورية، أو الانقطاعات، أو السلوك المحلي مقابل السلوك العالمي؟ تكمن حداثة هذا العمل في إطاره التكاملي الذي يقارن بين التقريبات المحلية (متسلسلات تايلور/القوى) والتمثيلات الدورية العالمية (متسلسلات فورييه)، مما يربط صراحةً بين النظرية الرياضية والممارسة الهندسية. تُظهر هذه الدراسة البحثية أنه بينما تتفوق متسلسلة تايلور في النمذجة المحلية عالية الدقة بالقرب من النقاط التحليلية، إلا أنها تفشل في تمثيل الأنظمة المتقطعة أو الدورية الشائعة في الاهتزازات الميكانيكية، ومعالجة الإشارات، والديناميكيات الحرارية. في المقابل، تلتقط متسلسلة فورييه هذه السلوكيات بدقة من خلال التحليل التوافقي، على الرغم من إظهارها ظاهرة جيبس بالقرب من الانقطاعات. يُسهم هذا البحث بشكل مباشر في المجال الهندسي من خلال توفير تصنيف موجه نحو اتخاذ القرار، يُرشد الممارسين في اختيار التوسيع الأمثل للمتسلسلات لحل مشاكل تحليل الاهتزازات، ونقل الحرارة، وتشخيص الأعطال، والصوتيات، وأنظمة التحكم. من خلال توضيح قيود التقارب، وقواعد التشغيل (مثل التفاضل والتكامل على أساس المصطلحات)، وقابلية التفسير الفيزيائي للمكونات التوافقية، تُعزز هذه المراجعة كلاً من الوضوح التربوي والموثوقية الحسابية في النمذجة الهندسية. في نهاية المطاف، يربط هذا العمل بين النظرية الرياضية المجردة والتصميم الهندسي التطبيقي، مما يُمكّن من استخدام أكثر وعياً وفعالية للأساليب القائمة على المتسلسلة في التحديات التكنولوجية الحديثة.

الكلمات المفتاحية: المتسلسلة الحسابية، المتسلسلة الهندسية، متسلسلة القوى، متسلسلة فورييه، متسلسلة تايلور، التقارب، الدوال الدورية، توسيع المتسلسلة، التحليل الرياضي، معالجة الإشارة.

1. Introduction

Series play a crucial role in mathematics, providing a foundation for understanding convergence, divergence, and various mathematical approximations [1]. This paper reviews the main types of series equations, their key powers, and how to solve these equations step by step, focusing on their mathematical calculation [2].

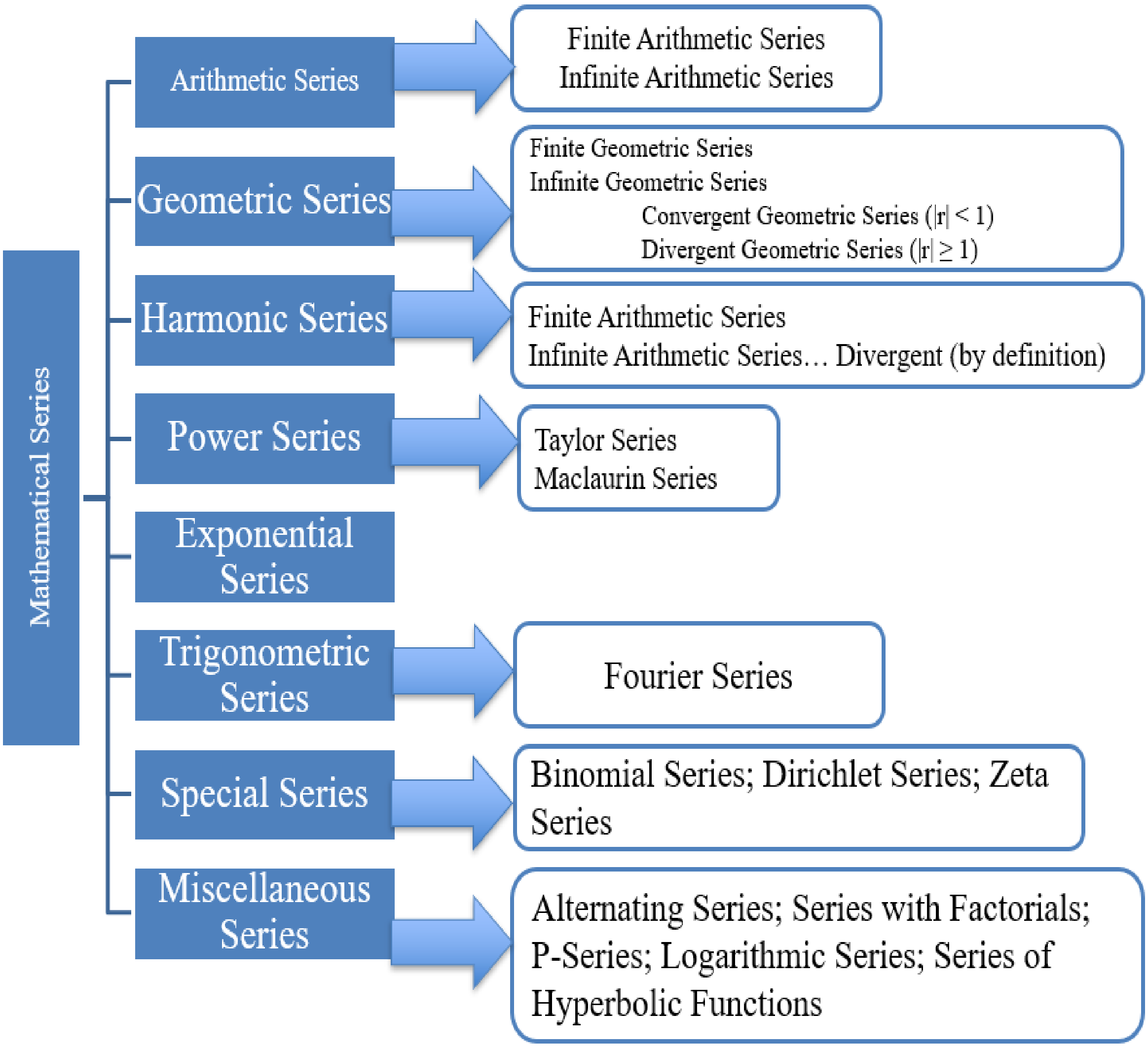

Figure.1. The hierarchical diagram of series types

2. Type of Series

2.1. Arithmetic Series

An arithmetic series is the sum of terms of an arithmetic sequence where each term differs from the previous one by a constant difference d. An arithmetic series is the sum of the terms in an arithmetic sequence, where each term increases (or decreases) by a constant difference [2], [3], [4], [5], [6], [7], [8]. In an arithmetic sequence, the difference between consecutive terms is called the common difference, denoted by "d".

General Form of an Arithmetic Sequence

The general form of an arithmetic sequence is [2], [3], [4], [5], [6], [7], [8]:

Where: a is the first term of the sequence. d is the common difference between consecutive terms

$$a, a+d, a+2d, a+3d,$$

$$\leq \frac{n}{2} \times \{2n + \{n - 1 \mid d\}$$

Arithmetic Series Formula: The sum of the first **n** terms of an arithmetic sequence is called an arithmetic series [2], [3], [4], [5], [6], [7], [8]. The formula to find the sum of the first **n** terms S_n is:

$$a_n = a_1 + (n-1)\mathrm{d}$$
$$S_n = \frac{n}{2} \times (2a + (n-1)\mathrm{d})$$

Alternatively, it can also be written as:

$$S_n = \frac{n}{2} \times (a + \mathrm{l})$$

Where:

- **n** is the number of terms.
- **a** is the first term.
- **l** is the last term of the series.
- a_n is the n-th term.
- a_1 is the first term.
- d is the common difference.
- n is the number of terms.

Sum of an Arithmetic Series: The sum of the first n terms (S_n) is given by [2], [3], [4]:

$$S_n = \frac{n}{2}(2a_1 + (n-1)d)$$

This formula allows for the efficient calculation of the sum of large sequences.

Application

Arithmetic series are often used in situations where there is a constant rate of change, such as calculating the total distance traveled by an object moving at a constant acceleration or finding the total of regularly spaced payments over time **[2], [3], [4], [5], [6], [7], [8]**.

2. 2. Geometric Series

A geometric series is a sequence of numbers where each term after the first is found by multiplying the previous term by a fixed, non-zero number called the "common ratio." Formula: The n-th term of a geometric series can be calculated using:

$$a_n = a_1 \cdot r^{(n-1)}$$

Where:

- a_n is the n-th term.
- a_1 is the first term.
- r is the common ratio.
- n is the number of terms.
- Sum of a Finite Geometric Series: The sum of the first n terms (S_n) is given by:

$$S_n = a_1 \frac{1 - r^n}{1 - r}$$

if $r \neq 1$.

- Sum of an Infinite Geometric Series: For $|r| < 1$, the sum of an infinite geometric series is:

$$S = \frac{a_1}{1 - r}$$

This provides a means to sum infinite sequences that converge.

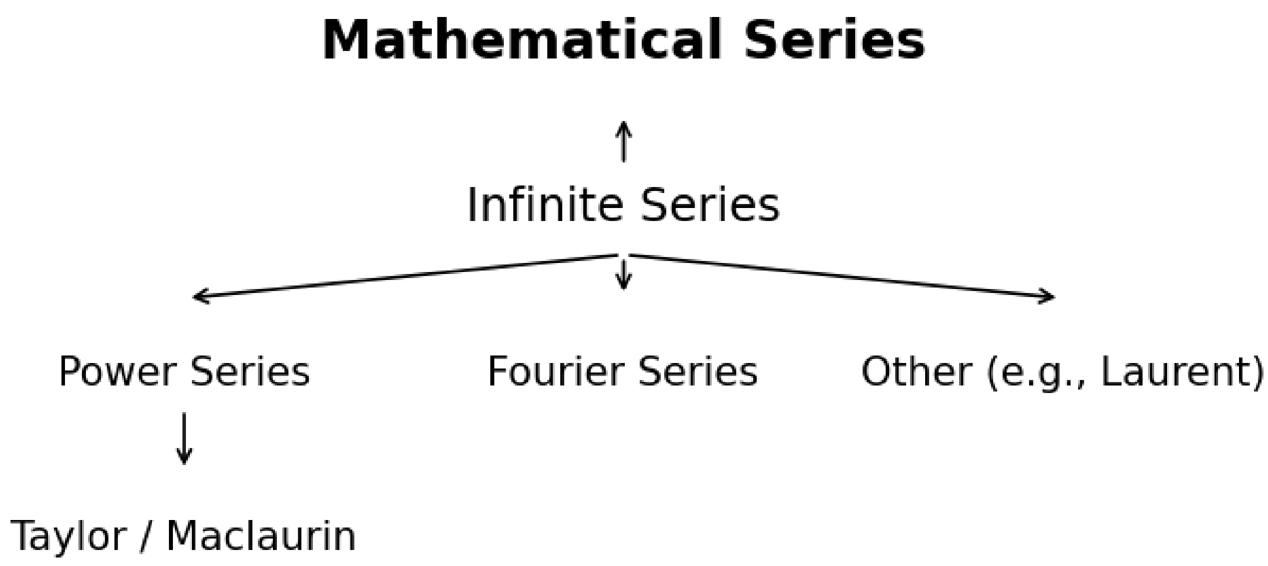

Figure 2. Hierarchical Diagram of Series Types

In mathematics, a geometric series is a series summing the terms of an infinite geometric sequence, in which the ratio of consecutive terms is constant. For example, the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ ……………………… is a geometric series with common ratio $\frac{1}{2}$, which converges to the sum of 1. Each term in a geometric series is the geometric mean of the term before it and the term after it, in the same way that each term of an arithmetic series is the arithmetic mean of its neighbors [2], [3], [4], [5], [6], [7], [8].

General Formula

The geometric series formula refers to the formula that gives the sum of a finite geometric sequence, the sum of an infinite geometric series, and the nth term of a geometric sequence. The sequence is of the form {a, ar, ar^2, ar^3, …….} where, a is the first term, and r is the "common ratio".

n^{th} term of the a geometric sequence a, ar, ar^2, ar^3, …….is

$$a_n = ar^{2n-1}$$

Sum of n term of a finite a geometric sequence is a

a, ar^2, ar^3, ……. ar^{n-1}

$=\frac{a(1-r^2)}{1-r}$ OR $=\frac{a(r^2-1)}{r-1}$ where $r\neq1$

Sum of infinite a geometric sequence is a, ar^2, ar^3, ……. $=\frac{a}{1-r}$, where $r < 1$

Geometric series

The geometric series formula

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^1 = 1+x+x^2+x^3+\cdots$$

Which is valid for $|x| < 1$, is one of the most important wamples of a power serias, as ane the exponential function formula [9], [10], [11].

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

and the sine formula

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n a^{3n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Valid for all real x. These power series are examples of Taylor series (or, more specifically, of Maclaurin series) [20], [21], [22]. , $x^{-1} + 1 + x^1 + x^2 + \cdots$ is not considarod a power. Negative powers are not permitted in an ordinary power series [12], [13], [14]; for instance, $x^{-1} + 1 + x^1 + x^2 + \cdots$ is not considered a power series (although it is a Laurent series) [9], [10], [11]. Similarly, fractional powers such as $x^{\frac{1}{2}}$ are not permitted; fractional powers arise in Puiseux series. The coefficients a_n must not depend on x, thus for instance $\sin(x)x + \sin(2x)x^2 + \sin(3x)x^3 + \cdots$ is not a power series.

Example a geometric sequence: Find all terms between $a_1 = -5$ and $a_4 = -135$ of a geometric sequence [9]. In other words, find all geometric means between the 1^{st} and 4^{th} terms.

Solution

Begin by finding the common ratio r. In this case, we are given the first and fourth terms:

$$\begin{aligned} a_n &= a_1 r^{n-1} \text{ Use } n = 4 \\ a_4 &= a_1 r^{4-1} \\ a_4 &= a_1 r^3 \end{aligned}$$

Substitute $a_1 = -5$ and $a_4 = -135$ into the above equation and then solve for r.

$$\begin{aligned} -135 &= -5r^3 \\ 27 &= r^3 \\ 3 &= r \end{aligned}$$

Next use the first term $a_1 = -5$ and the common ratio $r = 3$ to find an equation for the nth term of the sequence.

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_n &= -5(3)^{n-1} \end{aligned}$$

Now we can use $a_n = -5(3)^{n-1}$ where n is a positive integer to determine the missing terms.

$$\left.\begin{aligned} a_1 &= -5(3)^{1-1} = -5 \cdot 3^0 = -5 \\ a_2 &= -5(3)^{2-1} = -5 \cdot 3^1 = -15 \\ a_3 &= -5(3)^{3-1} = -5 \cdot 3^2 = -45 \\ a_4 &= -5(3)^{4-1} = -5 \cdot 3^3 = -135 \end{aligned}\right\} \text{geometric means}$$

Answer:

$$-15, -45$$

2.3. Power Series

In mathematics, a power series (in one variable) is an infinite series of the form

$$\sum_{n=0}^{\infty} \alpha_n (x-c)^2 = a_0 + \alpha_1(z-c) + a_2(x-c)^2 + \cdots$$

Where n an represents the coefficient of the nth term and c is a constant called the center of the series. Power series are useful in mathematical analysis, where they arise as Taylor series of infinitely differentiable functions [12], [13], [14]. In fact, Borel's theorem implies that every power series is the Taylor series of some smooth function. In many situations, the center c is equal to zero, for instance for Maclaurin series. In such cases, the power series takes the simpler

$$\sum_{n=0}^{\infty} \alpha_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

The partial sums of a power series are polynomials [12], [13], [14], the partial sums of the Taylor series of an analytic function are a sequence of converging polynomial approximations to the function at the center [20], [21], [22], and a converging power series can be seen as a kind of generalized polynomial with infinitely many terms. Conversely, every polynomial is a power series with only finitely many non-zero terms. Beyond their role in mathematical analysis, power series also occur in combinatorics as generating functions (a kind of formal power series) and in electronic engineering (under the name of the Z-transform). The familiar decimal notation for real numbers can also be viewed as an example of a power series, with integer coefficients, but with the argument x fixed at 1⁄10. In number theory, the concept of p-adic numbers is also closely related to that of a power series [12], [13], [14].

Example based on the Polynomial

Every polynomial of degree d can be expressed as a power series around any center c [12], [13], [14], where all terms of degree higher than d have a coefficient of zero.[1] For instance, the polynomial

$f(x) = x^2 + 2z + 3$ can be written as a power series around the center c=0 as

$$f(x) = 3 + 2x + 1x^2 + 0x^3 + 0x^4 + \cdots$$

or around the center c=1 as

$$f(x) = 6 + 4(x-1) + 1(x-1)^2 + 0(x-1)^3 + 0(x-1)^4 + \cdots$$

One can view power series as being like "polynomials of infinite degree", although power series are

Radius of convergence

A power series $\sum_{n=0}^{\infty} = a_n(x-c)^{\mathrm{n}}$ is convargent for some veluas of the variable x, which will always include a c since not the only poirt af convergence [12], [13], [14], then there is alwajys a number r with $0 < r \leq \infty$ such that the sariss convarges whenwer givan as

$$r = \liminf_{n \to -\infty} |a_{\mathrm{e}}|^{-\frac{1}{n}}$$

α, coulivalontly,

$$r^{-1} - \limsup_{\mathrm{n} \to \infty} |a_n|^{\frac{1}{n}}$$

$$r^{-1} = \lim_{n \to \infty} \left| \frac{\alpha_{n+1}}{a_n} \right|$$

The set of the complex numbers such that

Step-by-Step Solution for Convergence:

1. Determine the radius of convergence R using the formula:

R = 1 / limsup_{n -> ∞} sqrt[n]{|a_n|}.

2. Test the endpoints of the interval (-R, R) to check for convergence or divergence.

Operations on power series and addition and subtraction

When two functions f and g are decomposed into power series around the same center c, the power series of the sum or difference of the functions can be obtained by termwise addition and subtraction [12], [13], [14]. That is, if

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$$

And

$$g(x) = \sum_{n=0}^{\infty} b_n(x-c)^n$$

Then

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-c)^n$$

The sum of two power series will have a radius of convergence of at least the smaller of the two radii of convergence of the two series [12], [13], [14], [2] but possibly larger than either of the two. For instance it is not true that if two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ have the same radius of convergence, then $\sum_{n=0}^{\infty} (a_n + b_n) x^n$ also has this radius of convergence: if $a_n = (-1)^n$ and $b_n = (-1)^{n+1}\left(1 - \frac{1}{3^n}\right)$, for instance, then both series have the same radius of convergence of 1 , but the series $\sum_{n=0}^{\infty} (a_n + b_n) x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} x^n$ has a radius of convergence of 3.

Properties of Power Series

The power series is a special and extremely useful type of infinite series, and as illustrated in the preceding subsection, may be constructed by the Maclaurin formula, Eq. (1.44). However obtained [12], [13], [14], it will be of the general form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n,$$

where the coefficients a_i are constants, independent of x.

Equation (1.54) may readily be tested for convergence either by the Cauchy root test or the d'Alembert ratio test. If

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = R^{-1}$$

The series converges for $-R < x < R$. This is the interval or radius of convergence. Since the root and ratio tests fail when x is at the limit points $\pm R$, these points require special attention. For instance, if $a_n = n^{-1}$, then $R = 1$ and from Section 1.1 we can conclude that the series converges for $x = -1$ but diverges for $x = +1$. If $a_n = n!$, then $R = 0$ and the series diverges for all $x \neq 0$. Suppose power series has been found convergent for $-R < x < R$; then it will be uniformly and absolutely convergent in any interior interval $-S \leq x \leq S$, where $0 < S < R$. This may be proved directly by the Weierstrass M test. Since each of the terms $u_n(x) = a_n x^n$ is a continuous function of x and $f(x) = \sum a_n x^n$ converges uniformly for $-S \leq x \leq S, f(x)$ must be a continuous function in the interval of uniform convergence [12], [13], [14]. This behavior is to be contrasted with the strikingly different behavior of series in trigonometric functions, which are used frequently to represent discontinuous functions such as sawtooth and square waves. The new factors introduced by differentiation or integration do not affect either the root or the ratio test. Therefore, the power series may be differentiated or integrated as often as desired within the interval of uniform convergence.

Inversion of Power Series

Suppose we are given a series

$$y - y_0 = a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=1}^{\infty} a_n (x - x_0)^n.$$

This gives $(y - y_0)$ in terms of $(x - x_0)$. [12], [13], [14] However, it may be desirable to have an explicit expression for $(x - x_0)$ in terms of $(y - y_0)$. That is, we want an expression of the form

$$x - x_0 = \sum_{n=1}^{\infty} b_n (y - y_0)^n,$$

With the b_n to be determined in terms of the assumed known a_n. A brute-force approach, which is perfectly adequate for the first few coefficients, is simply to substitute Eq. (1.61) into Eq. (1.62). By equating coefficients of $(x - x_0)^n$ on both sides of Eq. (1.62), and using the fact that the power series is unique, we find

$$b_1 = \frac{1}{a_1}$$

$$b_2 = -\frac{a_2}{a_1^3}$$

$$b_3 = \frac{1}{a_1^5}(2a_2^2 - a_1 a_3)$$

$$b_4 = \frac{1}{a_1^7}(5a_1 a_2 a_3 - a_1^2 a_4 - 5a_2^3), \text{ and so on.}$$

Some of the higher coefficients are listed by Dwight. [4] A more general and much more elegant approach is developed by the use of complex variables in the first and second editions of Mathematical Methods for Physicists [12], [13], [14].

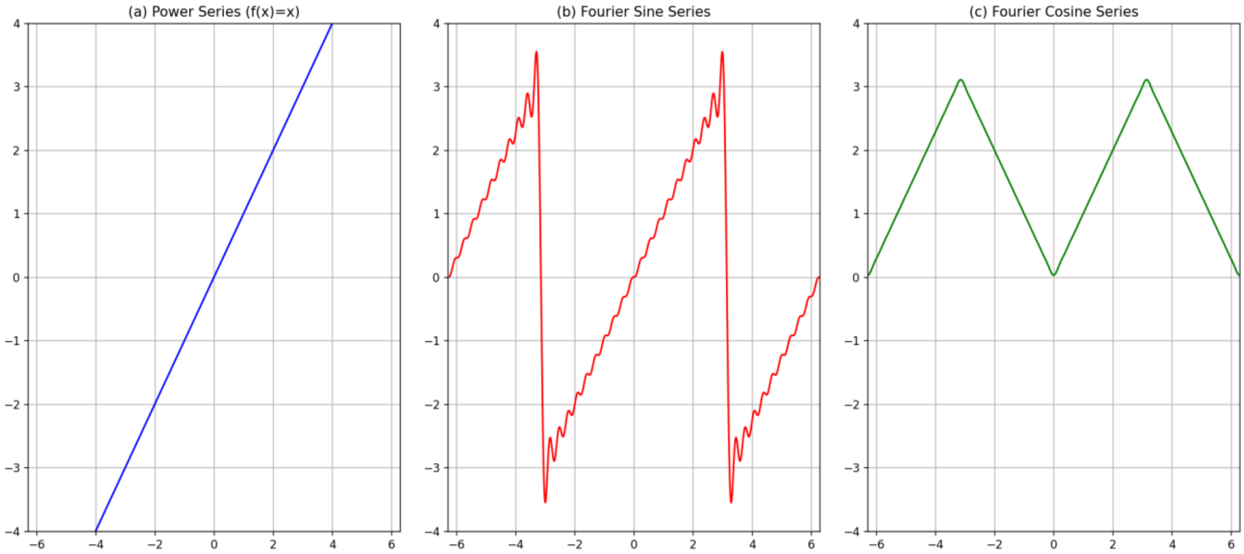

Figure 3 the distinct global behaviors of three different series expansions for the function f(x) = x, defined on the interval [0, π], when extended beyond their domain of definition (Generated by Python code).

Figure 3 above illustrates Panel (a) displays the power series expansion, which is identical to the original linear function f(x) = x and extends indefinitely as a straight line, reflecting its local nature centered at a point. In contrast, panel (b) shows the Fourier sine series, which produces an odd, periodic extension of the function with period 2π, characterized by Gibbs phenomenon near the discontinuities at integer multiples of π. Finally, panel (c) depicts the Fourier cosine series, resulting in an even, periodic extension with period 2π, forming a triangular wave pattern that converges to the average value at the points of discontinuity.

2.4. Fourier Series

Periodic phenomena involving waves, rotating machines (harmonic motion), or other repetitive driving forces are described by periodic functions [15]. Fourier series are a basic tool for solving ordinary differential equations (ODEs) and partial differential equations (PDEs) with periodic boundary conditions [15], [16], [17], [18], [19]. Fourier series, a powerful mathematical tool, enables the decomposition of complex periodic functions into simpler sine and cosine components. Developed by Jean-Baptiste Joseph Fourier in the early 19th century, this method is widely used across various engineering disciplines, including mechanical engineering. The Fourier series' ability to analyze and model periodic phenomena makes it invaluable for solving complex mechanical problems [15], [16], [17].

General Properties

A Fourier series is defined as an expansion of a function or representation of a function in a series of sines and cosines, such as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\, nx+\sum_{n=1}^{\infty} b_n\sin\, nx$$

The coefficients a_0, a_n, and b_n are related to $f(x)$ by definite integrals:

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(s)\cos\, nsds,\ n=0,1,2,\ldots,$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(s)\sin\, nsds,\ n=1,2,\ldots,$$

Which are subject to the requirement that the integrals exist. Note that a_0 is singled out for special treatment by the inclusion of the factor $\frac{1}{2}$. This is done so that Eq. (19.2) will apply to all $a_n, n=0$ as well as $n>0$. The conditions imposed on $f(x)$ to make Eq. (19.1) valid are that $f(x)$ have only a finite number of finite discontinuities and only a finite number of extreme values (maxima and minima) in the interval $[0,2\pi]$. [1] Expressing cos x and sin x in exponential form, we may rewrite Eq. (19.1) as

$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx},$ (19.4)

$c_n=\frac{1}{2}(a_n-ib_n),\ c_{-n}=\frac{1}{2}(a_n+ib_n),\ n>0$ (19.5)

$c_0=\frac{1}{2}a_0.$ (19.6)

Sturm-Liouville Theory

The ODE

$$-y''(x)=\lambda y(x)$$

on the interval $[0,2\pi]$ with boundary conditions $y(0)=y(2\pi), y'(0)=y'(2\pi)$ is a Sturm-Liouville problem, and these boundary conditions make it Hermitian. Therefore its eigenfunctions, either $\cos\, nx (n=0,1,\ldots)$ and $\sin\, nx (n=1,2,\ldots)$, or $\exp(inx)$ ($n=\cdots,-1,0,1,\ldots$), form a complete set, with eigenfunctions of different eigenvalues orthogonal. [15], [16], [17] Since the eigenfunctions have respective values n^2, those of different $|n|$ will automatically be orthogonal, while those of the same $|n|$ can be orthogonalized if necessary. Defining the scalar product for this problem as below

$$\langle f \mid g\rangle=\int_0^{2\pi} f^*(x)g(x)dx$$

It is easy to check that $\langle e^{inx} \mid e^{-inx}\rangle=0$ for $n\neq 0$, and if we write $\cos\, nx$ and $\sin\, nx$ as complex exponentials, it is also easy to see that $\langle \sin\, nx \mid \cos\, nx\rangle=0$. To make the eigenfunctions normalized, a simple approach is to note that the average value of $\sin^2\, nx$ or $\cos^2\, nx$ over an integer number of oscillations is $1/2$ (again for $n\neq 0$) [18], [19], so

$$\int_0^{2\pi}\sin^2\, nxdx=\int_0^{2\pi}\cos^2\, nxdx=\pi\ (n\neq 0),$$

and $\langle e^{inx} \mid e^{inx}\rangle=2\pi$.

The relationships identified above indicate that the eigenfunctions $\varphi_n=e^{inx}/\sqrt{2\pi}, (n=\ldots,-1,0,1,\ldots)$ form an orthonormal set, as do

$$\varphi_0 = \frac{1}{\sqrt{2\pi}},\ \varphi_n = \frac{\cos\ nx}{\sqrt{\pi}},\ \varphi_{-n} = \frac{\sin\ nx}{\sqrt{\pi}},\ (n = 1,2,\dots),$$

So expansions in these functions have the forms given in Eqs. (19.1) to (19.3) or Eqs. (19.4) to (19.6). Sturm-Liouville operator form a complete set, we know that our Fourier-series expansions of L2 functions will at least converge in the mean [15], [16], [17].

Discontinuous Functions

There are significant differences between the behavior of Fourier- and power-series expansions. A power series is essentially an expansion about a point, using only information from that point about the function to be expanded (including, of course, the values of its derivatives). We already know that such expansions only converge within a radius of convergence defined by the position of the nearest singularity [18], [19]. However, a Fourier series (or any expansion in orthogonal functions) uses information from the entire expansion interval, and therefore can describe functions that have “nonpathological” singularities within that interval. However, we also know that the representation of a function by an orthogonal expansion is only guaranteed to converge in the mean [15], [16], [17], [18], [19]. This feature comes into play for the expansion of functions with discontinuities, where there is no unique value to which the expansion must converge. However, for Fourier series, it can be shown that if a function f (x) satisfying the Dirichlet conditions is discontinuous at a point x0, its Fourier series evaluated at that point will be the arithmetic average of the limits of the left and right approaches:

$$f_{\text{Fourier series}}\ (x_0) = \lim_{\varepsilon\to 0}\left[\frac{f(x_0+\varepsilon)+f(x_0-\varepsilon)}{2}\right].$$

For proof of Eq. (19.7), see Jeffreys and Jeffreys or Carslaw (Additional Readings). It can also be shown that if the function to be expanded is continuous but has a finite discontinuity in its first derivative, its Fourier series will then exhibit uniform convergence (see Churchill, Additional Readings). These features make Fourier expansions useful for functions with a variety of types of discontinuities.

Example based SaWtooth Wave

An idea of the convergence of a Fourier series and the error in using only a finite number of terms in the series may be obtained by considering the expansion of

$$f(x) = \begin{cases} x, & 0 \le x < \pi \\ x - 2\pi, & \pi < x \le 2\pi \end{cases}$$

This is a sawtooth wave form, as shown in Fig. 19.1. Using Eqs. (19.2) and (19.3), we find the expansion to be

$$f(x) = 2\left[\sin\ x - \frac{\sin\ 2x}{2} + \frac{\sin\ 3x}{3} - \cdots + (-1)^{n+1}\frac{\sin\ nx}{n} + \cdots\right].$$

Figure 19.2 shows $f(x)$ for $0 \le x < 2\pi$ for the sum of 4,6 , and 10 terms of the series. Three features deserve comment.

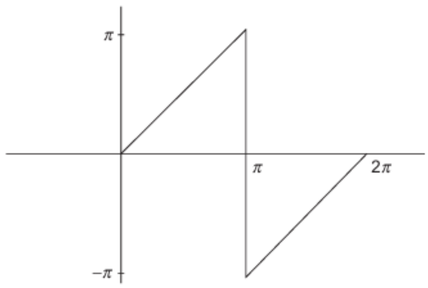

FIGURE.4. Sawtooth wave form [2]

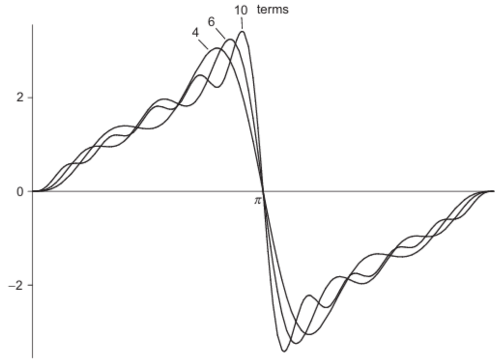

FIGURE.5. Expansion of sawtooth wave form, range [0,2π] [3]

1 There is a steady increase in the accuracy of the representation as the number of terms included is increased.
2 At $x = \pi$, where $f(x)$ changes discontinuously from $+\pi$ to $-\pi$, all the curves pass through the average of these two values, namely $f(\pi) = 0$.
3 In the vicinity of the discontinuity at $x = \pi$, there is an overshoot that persists and shows no sign of diminishing.

As a matter of incidental interest, setting $x = \pi/2$ in Eq. (19.9) leads to

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} = 2\left[1 - 0 - \frac{1}{3} - 0 + \frac{1}{5} - 0 - \frac{1}{7} + \cdots\right],$$

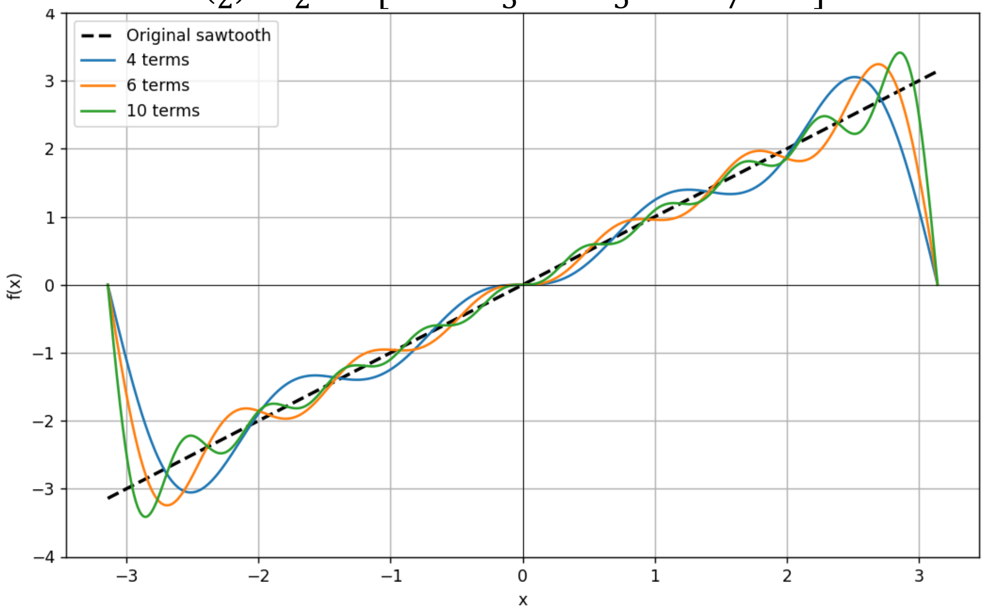

FIGURE 6 Fourier series of sawtooth wave generated by the researcher by using Python code.

Figure 6 above illustrates the convergence of the Fourier sine series for a sawtooth wave defined on the interval [0, π], demonstrating the progressive approximation achieved by including more terms in the series expansion. This Figure confirms the theoretical prediction that at the point of discontinuity (x = π), all partial sums converge to the arithmetic mean of the left- and right-hand limits, which is zero. Furthermore, the figure clearly exhibits the Gibbs phenomenon, characterized by the persistent overshoot near the discontinuities that does not diminish with an increasing number of terms, a fundamental property of Fourier series expansions for piecewise smooth functions.

19.1 General Properties

Yielding an alternate derivation of Leibniz's formula for $\pi/4$, which was obtained by another method in Exercise 1.3.2. Fourier series are used extensively to represent periodic functions, especially wave forms for signal processing [15], [16], [17], [18], [19]. The form of the series is inherently periodic; the expansions in Eqs. (19.1) and (19.4) are periodic with period 2π, with $\sin nx, \cos nx$, and $\exp(inx)$, each completing n cycles of oscillation in that interval. Thus, while the coefficients in a Fourier expansion are determined from an interval of length 2π, the expansion itself (if the function involved is actually periodic) applies for an indefinite range of x. The periodicity also means that the interval used for determining the coefficients need not be $[0,2\pi]$ but may be any other interval of that length. Often one encounters situations in which the formulas in Eqs. (19.2) and (19.3) are changed so that their integrations run between $-\pi$ and π. In fact, it would have been natural to have restated [15], [16], [17], [18], [19]. Example 19.1.1 as dealing with $f(x)=x$, for $-\pi<x<\pi$. This of course does not remove the discontinuity or change the form of the Fourier series. The discontinuity has simply been moved to the ends of the interval in x. The natural interval for a Fourier expansion will be the wavelength of our wave form, so it may make sense to redefine our Fourier series so that Eq. (19.1)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}+\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

with

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(s)\cos \frac{n\pi s}{L} ds,\ n=0,1,2,\ldots, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(s)\sin \frac{n\pi s}{L} ds,\ n=1,2,\ldots \end{aligned}$$

In many problems the x dependence of a Fourier expansion describes the spatial dependence of a wave distribution that is moving (say, toward $+x$) with phase velocity v. This means that in place of x we need to write $x-vt$, and this substitution carries the implicit assumption that the wave form retains the same shape as it moves forward [15], [16], [17], [18], [19]. [2] The individual terms of the Fourier expansion can now be given an interesting interpretation. Taking as an example the term

$$\cos\left[\frac{n\pi}{L}(x-vt)\right]$$

It describes a contribution of wavelength $2L/n$ (when x increases this much at constant t, the argument of the cosine function increases by 2π). We also note that the period of the oscillation (the change in t at constant x for one cycle of the cosine function) is $T=2L/nv$, corresponding to the oscillation frequency $\nu=nv/2L$. If we call the frequency for $n=1$ the fundamental frequency and denote it $\nu_0=v/2L$, we identify the terms for each $n>1$ in the Fourier series as describing overtones, or harmonics of the fundamental frequency, with individual frequencies $n\nu_0$. A typical problem for which Fourier analysis is suitable is one in which a particle undergoing oscillatory motion is subject to a periodic driving force [15], [16], [17], [18], [19]. If the problem is described . A typical problem for which Fourier analysis is suitable is one in which a particle under going oscillatory motion is subject to a periodic driving force. If the problem is described by a linear ODE, we may make a Fourier expansion of the driving force and solve for each harmonic individually [15], [16], [17], [18], [19]. This makes the Fourier expansion a practical tool as well as a nice analytical device. The stress, however, that its utility depends crucially on the linearity of our problem; in nonlinear problems an overall solution is not a superposition of component solutions [15], [16], [17].

Example based on the Different Expansions of $f(x) = x$
This research consider three possible ways to expand $f(x) = x$ based on its values on the range $[0, \pi]$:

- Its power-series expansion will (obviously) have the power-series expansion $f(x) = x$.
- Comparing with Example 19.1.1, its Fourier sine series will have the form given in Eq. (19.9).
- Its Fourier cosine series will have coefficients determined from

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos\, nx dx = \begin{cases} \pi, & n = 0, \\ -\frac{4}{n^2\pi}, & n = 1,3,5,\dots, \\ 0, & n = 2,4,6,\dots, \end{cases}$$

corresponding to the expansion

$$f(x) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{4}{\pi}\frac{\cos\,(2n+1)x}{(2n+1)^2}$$

All three of these expansions represent $f(x)$ well in the range of definition [15], [16], [17], $[0, \pi]$, but their behavior becomes strikingly different outside that range. We compare the three expansions for a range larger than $[0, \pi]$ in Figure. 4 below.

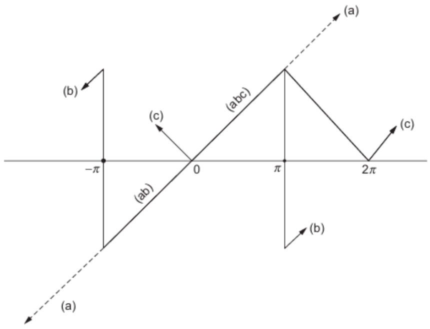

FIGURE 7 Expansions of f(x)=x on [0,π]:(a) power series,(b) Fourier sine series, (c) Fourier cosine series.[6]

Operations on Fourier Series

Term-by-term integration of the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\, nx + \sum_{n=1}^{\infty} b_n\sin\, nx \qquad (19.15)$$

yields

$$\int_{x_0}^{x} f(x)dx = \frac{a_0 x}{2}\Big|_{x_0}^{x} + \sum_{n=1}^{\infty} \frac{a_n}{n}\sin\, nx\Big|_{x_0}^{x} - \sum_{n=1}^{\infty} \frac{b_n}{n}\cos\, nx\Big|_{x_0}^{x} \qquad (19.16)$$

Clearly, the effect of integration is to place an additional power of n in the denominator of each coefficient. This results in more rapid convergence than before [15], [16], [17], [18], [19]. Consequently, a convergent Fourier series may always be integrated term by term, the resulting series converging uniformly to the integral of the original function. Indeed, term-by-term integration may be valid even if the original series, Eq. (19.15), is not itself convergent. The function $f(x)$ need only be integrable. A discussion will be found in Jeffreys and Jeffreys (Additional Readings).
Strictly speaking, Eq. (19.16) may not be a Fourier series; that is, if $a_0 \neq 0$, there will be a term $\frac{1}{2}a_0 x$. However,

$$\int_{x_0}^{x} f(x)dx - \frac{1}{2}a_0 x \qquad (19.17)$$

will still be a Fourier series.
The situation regarding differentiation is quite different from that of integration. Here the word is caution. Consider the series for
$f(x) = x, \ -\pi < x < \pi.$ (19.18)
We readily found (in Example 19.1.1) that the Fourier series is
$x = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}, \ -\pi < x < \pi.$ (19.19)
Differentiating term by term, we obtain
$1 = 2\sum_{n=1}^{\infty} (-1)^{n+1} \cos nx$ (19.20)
which is not convergent. Warning: Check your derivative for convergence.
For the triangular wave shown in Fig. 19.4 (and treated in Exercise 19.2.9), the Fourier expansion is
$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,\text{odd}}^{\infty} \frac{\cos nx}{n^2},$ (19.21)
which converges more rapidly than the expansion of Eq. (19.19); in fact, it exhibits uniform convergence. Differentiating term by term we get

$$f'(x) = \frac{4}{\pi}\sum_{n=1,\text{ odd}}^{\infty} \frac{\sin nx}{n},$$

which is the Fourier expansion of a square wave, (19.22)

$$f'(x) = \begin{cases} 1, & 0 < x < \pi, \\ -1, & -\pi < x < 0. \end{cases}$$

Inspection of Fig. 19.3 verifies that this is indeed the derivative of our triangular wave. (19.23)

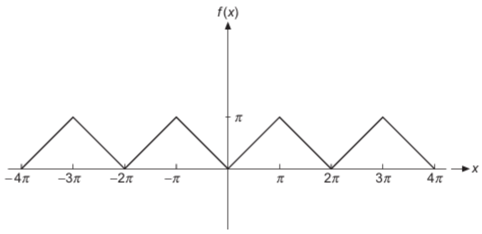

Figure 8 Triangular wave.

- As the inverse of integration, the operation of differentiation has placed an additional factor n in the numerator of each term. This reduces the rate of convergence and may, as in the first case mentioned, render the differentiated series divergent.
- In general, term-by-term differentiation is permissible if the series to be differentiated is uniformly convergent.

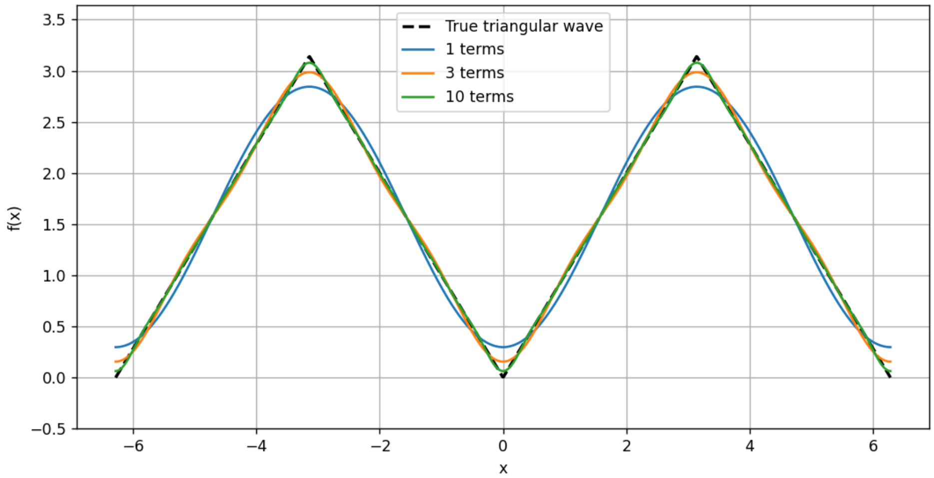

Figure 9 Triangular Wave and Fourier Series (Generated by Python code).

Figure 9 above as illustrates the Fourier series approximation of a triangular wave, demonstrating the progressive convergence achieved by including more harmonic terms in the series expansion. The dashed black line represents the true, continuous triangular waveform, while the solid colored lines depict the partial sums using 1, 3, and 10 terms, respectively, revealing a systematic improvement in accuracy as the number of harmonics increases. Unlike the sawtooth wave, this function is continuous but possesses discontinuities in its first derivative at the peaks and troughs; consequently, its Fourier series exhibits uniform convergence, meaning the approximation error diminishes uniformly across the entire domain as more terms are added. This visualization underscores the utility of Fourier series for representing piecewise-defined functions, where the global nature of the expansion allows it to capture the overall shape effectively, even with a relatively small number of terms.

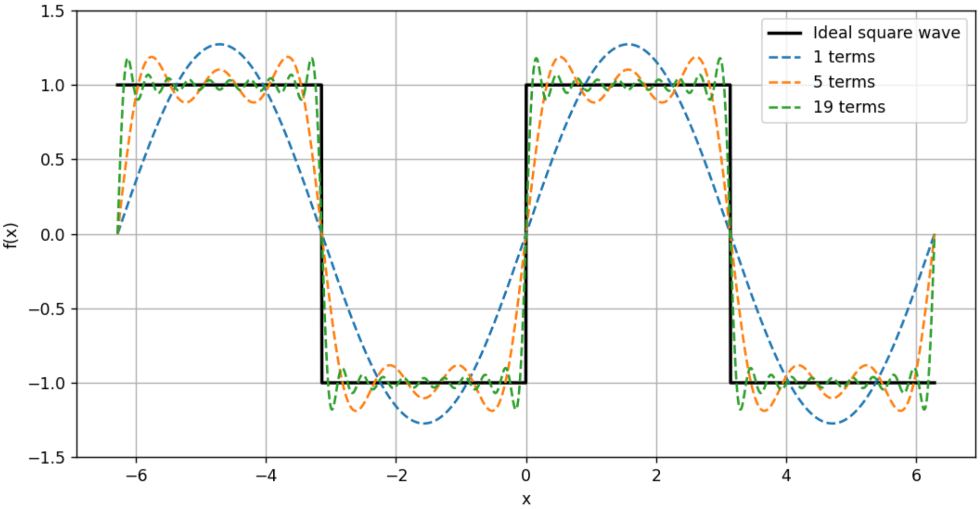

Figure 10 Supplementary: Square Wave Fourier Series (Derivative of Triangular) (generated by Python code)

Figure 10 above illustrates the Fourier series approximation of an ideal square wave, a classic example of a discontinuous periodic function. The black dashed line represents the true square wave, while the colored curves depict the partial sums using 1, 5, and 19 terms, demonstrating the progressive convergence towards the target waveform. This visualization explicitly confirms the theoretical prediction that at points of discontinuity (e.g., $x = 0, \pm 2\pi$, etc.), all partial sums converge to the arithmetic mean of the left- and right-hand limits, which is zero for this symmetric wave. Furthermore, the figure clearly exhibits the Gibbs phenomenon, characterized by the persistent overshoot near the discontinuities that does not diminish with an increasing number of terms, a fundamental property inherent to Fourier series expansions of piecewise smooth functions.

Example SUMMATION OF A FOURIER SERIES

Consider the series $\sum_{n=1}^{\infty}(1/n)\cos\ nx, x \in (0,2\pi)$. Since this series is only conditionally convergent (and diverges at $x = 0$), we take

$$\sum_{n=1}^{\infty}\frac{\cos\ nx}{n} = \lim_{r\to 1}\sum_{n=1}^{\infty}\frac{r^n\cos\ nx}{n}$$

absolutely convergent for $|r| < 1$. Our procedure is to try forming power series by transforming the trigonometric functions into exponential form:

$$\sum_{n=1}^{\infty}\frac{r^n\cos\ nx}{n} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{r^n e^{inx}}{n} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{r^n e^{-inx}}{n}$$

Now, these power series may be identified as Maclaurin expansions of $-\ln(1-z)$, with $z = re^{ix}$ or re^{-ix}. From Eq. (1.97),

$$\sum_{n=1}^{\infty}\frac{r^n\cos\ nx}{n} = -\frac{1}{2}\left[\ln\left(1-re^{ix}\right) + \ln\left(1-re^{-ix}\right)\right]$$

$$= -\ln\left[(1+r^2) - 2r\cos\ x\right]^{1/2}$$

Setting $r = 1$, we see that

$$\sum_{n=1}^{\infty}\frac{\cos\ nx}{n} = -\ln(2-2\cos\ x)^{1/2}$$

$$= -\ln\left(2\sin\frac{x}{2}\right),\ \left(0 < x < 2\pi\right) \qquad (9.24)$$

Both sides of this expression diverge as $x \to 0$ and as $x \to 2\pi$ [4]

2.5. Taylor Series

Taylor series of a function is an infinite sum of terms, that is expressed in terms of the function's derivatives at any single point, where each following term has a larger exponent like x, x^2, x^3, etc. Taylor series formula thus helps in the mathematical representation of the Taylor series [20], [21], [22].

Taylor Series Formula

The Taylor series formula helps to expand a function around a value of the variable using the derivatives of the function. It can be represented as [20], [21], [22],

$$f(x) = f(a) + f'(a)(x-a) + \left[\frac{f''(a)}{2!}(x-a)^2\right] + \left[\frac{f'''(a)}{3!}(x-a)^3\right] + \cdots .. + \left[\frac{f^{(n)}(a)}{n!}(x-a)^n\right]$$

OR

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}a}{n!}\times(x-a)^n$$

Here,

- $f(x)$ = Real or complex-valued function, that is infinitely

Taylor's Expansion

Taylor's expansion is a powerful tool for the generation of power series representations of functions. The derivation presented here provides not only the possibility of an expansion into a finite number of terms plus a remainder that may or may not be easy to evaluate, but also the possibility of the expression of a function as an infinite series of powers.
We assume that our function $f(x)$ has a continuous nth derivative [2] in the interval $a \leq x \leq b$. We integrate this nth derivative n times; the first three integrations yield

$$\int_a^x f^{(n)}(x_1)dx_1 = f^{(n-1)}(x_1)\Big|_a^x = f^{(n-1)}(x) - f^{(n-1)}(a),$$

$$\int_a^x dx_2 \int_a^{x_2} f^{(n)}(x_1)dx_1 = \int_a^x dx_2\left[f^{(n-1)}(x_2) - f^{(n-1)}(a)\right]$$
$$= f^{(n-2)}(x) - f^{(n-2)}(a) - (x-a)f^{(n-1)}(a),$$

$$\int_a^x dx_3 \int_a^{x_3} dx_2 \int_a^{x_2} f^{(n)}(x_1)dx_1 = f^{(n-3)}(x) - f^{(n-3)}(a)$$
$$- (x-a)f^{(n-2)}(a) - \frac{(x-a)^2}{2!}f^{(n-1)}(a).$$

Finally, after integrating for the nth time,

$$\int_a^x dx_n \cdots \int_a^{x_2} f^{(n)}(x_1)dx_1 = f(x) - f(a) - (x-a)f'(a) - \frac{(x-a)^2}{2!}f''(a)$$
$$- \cdots - \frac{(x-a)^{n-1}}{(n-1)!}f^{n-1}(a)$$

Note that this expression is exact. No terms have been dropped, no approximations made. Now, solving for $f(x)$, we have

$$f(x) = f(a) + (x-a)f'(a)$$
$$+\frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n, \qquad (1.40)$$

where the remainder, R_n, is given by the n-fold integral

$$R_n = \int_a^x dx_n \cdots \int_a^{x_2} dx_1 f^{(n)}(x_1) \qquad (1.41)$$

We may convert R_n into a perhaps more practical form by using the mean value theorem of integral calculus:
$\int_a^x g(x)dx = (x-a)g(\xi)$ (1.42) with $a \leq \xi \leq x$. By integrating n times we get the Lagrangian form [3] of the remainder:

$$R_n = \frac{(x-a)^n}{n!}f^{(n)}(\xi) \qquad (1.43)$$

With Taylor's expansion in this form, there are no questions of infinite series convergence. The series contains a finite number of terms, and the only questions concern the magnitude of the remainder. When the function $f(x)$ is such that $\lim_{n\to\infty} R_n = 0$, Eq. (1.40) becomes Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}f^{(n)}(a) \qquad (1.44)$$

The first time $n!$ with $n = 0$. Note that we define $0! = 1$.

Taylor series specifies the value of a function at one point, x, in terms of the value of the function and its derivatives at a reference point a. It is an expansion [20], [21], [22] in powers of the change in the variable, namely $x - a$. This idea can be emphasized by writing Taylor's series in an alternate form in which we replace x by $x + h$ and a by x :

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x).$$

Taylor series are often used in situations where the reference point, a, is assigned the value zero. In that case the expansion is referred to as a Maclaurin series, [20], [21], [22] and Eq. (1.40) becomes

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0).$$

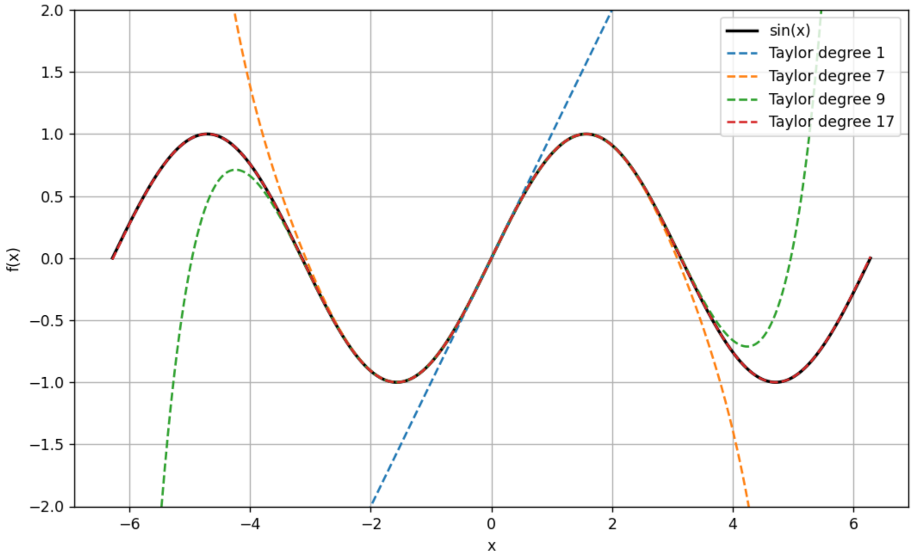

Figure 11 Taylor series approximations of sin(x) up to various degrees (Generated by Python code)

Figure 11 above illustrates the progressive approximation of the transcendental function f(x) = sin(x) using Taylor series expansions of increasing polynomial degree centered at x=0 (Maclaurin series). The black solid line represents the true sine function, while the dashed and dotted lines depict the approximations using 1st, 7th, 9th, and 17th-degree polynomials, demonstrating that higher-degree expansions provide a more accurate local representation over an increasingly wider interval. This visualization confirms the theoretical principle that a Taylor series converges to the original function within its radius of convergence, which for sin(x) is infinite, allowing for arbitrarily precise approximation by including sufficient terms. However, it also highlights a key limitation: lower-degree polynomials diverge significantly from the target function as |x| increases, underscoring that Taylor series are inherently local approximations whose accuracy diminishes rapidly outside the immediate neighborhood of the expansion point.

Example 1.2.3 EXPONENTIAL FUNCTION

Let $f(x) = e^x$. Differentiating, then setting $x = 0$, we have

$$f^{(n)}(0) = 1$$

for all $n, n = 1,2,3, \ldots$. Then, with Eq. (1.46), we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \qquad (1.47)$$

This is the series expansion of the exponential function. Some authors use this series to define the exponential function. Although this series is clearly convergent for all x, as may be verified using the d'Alembert ratio test, it is instructive to check the remainder term, R_n. By Eq. (1.43) we have

$$R_n = \frac{x^n}{n!} f^{(n)}(\xi) = \frac{x^n}{n!} e^{\xi},$$

where ξ is between 0 and x. Irrespective of the sign of x,

$$|R_n| \le \frac{|x|^n e^{|x|}}{n!}$$

No matter how large $|x|$ may be, a sufficient increase in n will cause the denominator of this form for R_n to dominate over the numerator, and $\lim_{n\to\infty} R_n = 0$. Thus, the Maclaurin expansion of e^x converges absolutely over the entire range $-\infty < x < \infty$.

Now that we have an expansion for exp (x), we can return to Eq. (1.45), and rewrite that equation in a form that focuses on its differential operator characteristics. Defining D as the operator d/dx, we have

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n \mathrm{D}^n}{n!} f(x) = e^{h\mathrm{D}} f(x). \qquad (1.48)$$

Taylor and Maclaurin Series

The Taylor series3of a function the complex analog of the real Taylor series is [20], [21], [22]

$$f(z) = \sum_{n=1}^{\infty} a_n (z-z_0)^n \quad \text{where} \quad a_n = \frac{1}{n!} f^{(n)}(z_0) \qquad (1)$$

Or, by (1), Sec. 14.4,

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}} dz^* \qquad (2)$$

In (2) we integrate counterclockwise around a simple closed path C that contains z_0 in its interior and is such that $f(z)$ is analytic in a domain containing C and every point inside C.

A Maclaurin series [3] is a Taylor series with center $z_0 = 0$.

[2] LEONARDO OF PISA, called FIBONACCI (= son of Bonaccio), about 1180-1250, Italian mathematician, credited with the first renaissance of mathematics on Christian soil.

The remainder of the Taylor series (1) after the term $a_n(z-z_0)^n$ is

$$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)} dz^* \qquad (3)$$

(proof below). Writing out the corresponding partial sum of (1), we thus have

$$f(z) = f(z_0) + \frac{z-z_0}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z-z_0)^n}{n!} f^{(n)}(z_0) + R_n(z) \qquad (4)$$

This is called Taylor's formula with remainder.

Taylor series are power series. From the last section, we know that power series represent analytic functions. The analytic function can be represented by a power series, namely [20], [21], [22], by Taylor series [20], [21], [22]. This makes the Taylor series very important in complex analysis. Indeed, they are more fundamental in complex analysis than their real counterparts are in calculus.

Taylor's Theorem

Let $f(z)$ be analytic in a domain D, and let $z = z_0$ be any point in D. Then there exists precisely one Taylor series (1) with center z_0 that represents $f(z)$. This representation is valid in the largest open disk with a center z_0 in which $f(z)$ is analytic [20], [21], [22]. The remainders $R_n(z)$ of (1) can be represented in the form (3). The coefficients satisfy the inequality

$$|a_n| \leqq \frac{M}{r^n}$$

Where M is the maximum of $|f(z)|$ on a circle $|z - z_0| = r$ in D whose interior is also in D.

PROOF. The key tool is Cauchy's integral formula in Sec. 14.3; writing z and z^* instead of z_0 and z (so that z^* is the variable of integration), we have

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z^* - z} dz^*$$

z lies inside C, for which we take a circle of radius r with center z_0 and interior in D (Fig. 367). We develop $1/(z^* - z)$ in (6) in powers of $z - z_0$. By a standard algebraic manipulation (worth remembering!) we first have

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0 - (z - z_0)} = \frac{1}{(z^* - z_0)\left(1 - \frac{z - z_0}{z^* - z_0}\right)}$$

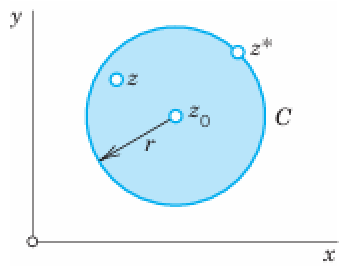

Figure 12 Cauchy formula [3].

For later use, this research review study note that since z^* is on C while z is inside C, this research review study have (7*) [1].

$$\left|\frac{z - z_0}{z^* - z_0}\right| < 1.$$

To (7) this research review study now apply the sum formula for a finite geometric sum (8*)

$$1 + q + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q} = \frac{1}{1 - q} - \frac{q^{n+1}}{1 - q} \quad (q \neq 1)$$

which this research review study use in the form (take the last term to the other side and interchange sides)

$$\frac{1}{1 - q} = 1 + q + \cdots + q^n + \frac{q^{n+1}}{1 - q}$$

Applying this with $q = (z - z_0)/(z^* - z_0)$ to the right side of (7), this research review study get

$$\frac{1}{z^* - z} = \frac{1}{z^* - z_0}\left[1 + \frac{z - z_0}{z^* - z_0} + \left(\frac{z - z_0}{z^* - z_0}\right)^2 + \cdots + \left(\frac{z - z_0}{z^* - z_0}\right)^n\right] + \frac{1}{z^* - z}\left(\frac{z - z_0}{z^* - z_0}\right)^{n+1}$$

This research review study insert this into (6). Powers of $z - z_0$ do not depend on the variable of integration z^*, so that we may take them out from under the integral sign [1]. This yields

$$f(z)=\frac{1}{2\pi i}\oint_C \frac{f(z^*)}{z^*-z_0}dz^* \quad +\frac{z-z_0}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^2}dz^*+\cdots$$
$$\cdots+\frac{(z-z_0)^n}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz^*+R_n(z)$$

with $R_n(z)$ given by (3). The integrals are those in (2) related to the derivatives, so that this research review study have proved the Taylor formula (4). Since analytic functions have derivatives of all orders, we can take n in (4) as large as we please. If we let n approach infinity, we obtain (1). Clearly, (1) will converge and represent $f(z)$ if and only if [1].

$$\lim_{n\to\infty} R_n(z)=0$$

This research review study prove (9) as follows. Since z^* lies on C, whereas z lies inside C (Fig. 367), we have $|z^*-z|>0$. Since $f(z)$ is analytic inside and on C, it is bounded, and so is the function $f(z^*)/(z^*-z)$, say,

$$\left|\frac{f(z^*)}{z^*-z}\right|\leqq \tilde{M}$$

For all z^* on C. Also, C has the radius $r=|z^*-z_0|$ and the length $2\pi r$. Hence by the ML-inequality (Sec. 14.1) this research review study obtain from (3)

$$|R_n| \quad =\frac{|z-z_0|^{n+1}}{2\pi}\left|\oint_C \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)}dz^*\right|$$
$$\leqq \frac{|z-z_0|^{n+1}}{2\pi}\tilde{M}\frac{1}{r^{n+1}}2\pi r=\tilde{M}\left|\frac{z-z_0}{r}\right|^{n+1}$$

Now $|z-z_0|<r$ because z lies inside C. Thus $|z-z_0|/r<1$, so that the right side approaches 0 as $n\to\infty$. This proves that the Taylor series converges and has the sum $f(z)$. Uniqueness follows from Theorem 2 in the last section [20], [21], [22], [23], [24], [25]. Finally, (5) follows from a_n in (1) and the Cauchy inequality in Sec. 14.4. This proves Taylor's theorem [1], [26], [27], [28]. Accuracy of Approximation. This research review study can achieve any preassinged accuracy in approximating $f(z)$ by a partial sum of (1) by choosing n large enough. This is the practical use of formula (9). [1], [29], [30], [31], [32], [33]. Radius of Convergence. On the circle of convergence of (1) there is at least one singular point of $f(z)$, that is, a point $z=c$ at which $f(z)$ is not analytic (but such that every disk with center c contains points at which $f(z)$ is analytic). This research review study also say that $f(z)$ is singular at c or has a singularity at c. Hence the radius of convergence R of (1) is usually equal to the distance from z_0 to the nearest singular point of $f(z)$. (Sometimes R can be greater than that distance: Lnz is singular on the negative real axis, whose distance from $z_0=-1+i$ is 1 , but the Taylor series of Ln z with center $z_0=-1+i$ has radius of convergence $\sqrt{2}$.) [1], [20], [21], [22].

Example: Evaluate the Taylor Series for $f(x)=\cos(x)$ for $x=0$.

Solution: this research review study need to take the derivatives of the cos x and evaluate them at $x=0$.

$$f(x)=\cos x\Rightarrow f(0)=1$$
$$f'(x)=-\sin x\Rightarrow f'(0)=0$$
$$f''(x)=-\cos x\Rightarrow f''(0)=-1$$
$$f'''(x)=\sin x\Rightarrow f'''(0)=0$$
$$f''''(x)=\cos x\Rightarrow f''''(4)=1$$
$$f(5)(x)=-\sin x\Rightarrow f(5)(0)=0$$
$$f(6)(x)=-\cos x\Rightarrow f(6)(0)=-1$$

Therefore, according to the Taylor series expansion;

$$\begin{aligned}\cos x &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \cdots \\ &= 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 - \frac{1}{6!}x^6 + \cdots \\ \cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}\end{aligned}$$

Example: Evaluate the Taylor Series for $f(x) = x^3 - 10x^2 + 6$ at $x = 3$.

Solution: First, we will find the derivatives of the given function.

$$\begin{aligned} &f(x) = x^3 - 10x^2 + 6 \Rightarrow f(3) = -57 \\ &f'(x) = 3x^2 - 20x \Rightarrow f'(3) = 33 \\ &f''(x) = 6x - 20 \Rightarrow f''(3) = -2 \\ &f'''(x) = 6 \Rightarrow f'''(3) = 6 \\ &f''''(x) = 0 \end{aligned}$$

Therefore, the required series is

$$\begin{aligned} x^3 - 10x^2 + 6 &= \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n \\ &= f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3 + 0 \\ &= -57 - 33(x-3) - (x-3)^2 + (x-3)^3 \end{aligned}$$

Example : Approximate $f(x) = \sin x$ using Taylor series up to polynomials of 1st, 7 th, 9 th, and 17 th degrees and compare the results on a graph for the interval $[-4\pi, 4\pi]$.
Solution [20], [21], [22], [34], [35], [36]:

$$\begin{aligned} &f^{(0)} = \sin x && f^{(0)}(0) = 0 \\ &f^{(1)} = \cos x && f^{(1)}(0) = 1 \\ &f^{(2)} = -\sin x && f^{(2)}(0) = 0 \\ &f^{(3)} = -\cos x && f^{(3)}(0) = -1 \\ &f^{(4)} = \sin x && f^{(4)}(0) = 0 \\ &f^{(5)} = \cos x && f^{(5)}(0) = 1 \\ &f^{(6)} = -\sin x && f^{(6)}(0) = 0 \end{aligned}$$

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$$\begin{aligned} &f^{(7)} = -\cos x && f^{(7)}(0) = -1 \\ &f^{(8)} = \sin x && f^{(8)}(0) = 0 \\ &f^{(9)} = \cos x && f^{(9)}(0) = 1 \\ &f^{(10)} = -\sin x && f^{(10)}(0) = 0 \\ &\vdots && \vdots \end{aligned}$$

Taylor series expansion for different orders are

$$f(x) = x + O(\Delta x^2)$$
$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + O(\Delta x^8)$$
$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + O(\Delta x^{10}).$$

While the approximated functions wander off to infinity, retaining sufficient amount of terms as

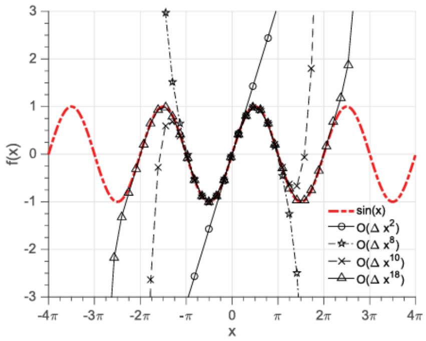

Figure 13 Plot of function sin x and its successive Taylor series expansions [20], [21], [22]. To closely represent the sine function around the origin ($x = 0$). Taylor series expansion up to the 17 th degree can recover the sine to closely represent the sine function around the origin (x =0) [20], [21], [22]. Taylor series expansion up to the 17th degree can recover the sine function accurately from −2π to 2π. If we were to retain more terms of the Taylor series expansion [20], [21], [22], [38], [39], [40], this research review study expect to recover the sine function accurately from −50π to 50π or −100π to 100π. By 81.215.11.52 on 12/09/24. Re-use and distribution is strictly not permitted. The expansion of all functions does not behave as well as the sine function. In fact, increasing the degree of expansion of function 1, 1−x will not help its divergence for |x| > 1.

Applications of Series

- Series have numerous applications in mathematics, physics, engineering, and other fields. Some common applications include:
- Calculus: Taylor and Maclaurin series for function approximation.
- Differential Equations: Power series solutions to differential equations.
- Probability and Statistics: Probability distributions and generating functions.
- Physics: Fourier series for periodic functions.
- Engineering: Signal processing and control systems.

Discussion

This literature review provides a structured and comparative analysis of five principal types of mathematical series arithmetic, geometric, power, Taylor, and Fourier highlighting their theoretical underpinnings, convergence characteristics, computational techniques, and domain-specific applications [1].The discussion that follows synthesizes key insights, contrasts methodological behaviors, and situates the findings within broader mathematical and applied contexts. First, the review underscores a fundamental dichotomy between local and global series representations [2], [3], [4]. Taylor (and Maclaurin) series, as instances of power series, are inherently local: they approximate a function in the neighborhood of a single point using derivative information. Their utility is constrained by the radius of convergence, which is dictated by the nearest singularity in the complex plane. In contrast, Fourier series offer a global representation over an entire interval, leveraging orthogonality and integral transforms to reconstruct periodic functions even those with discontinuities using sine and cosine bases. This

global nature enables Fourier series to model real-world phenomena like vibrations, heat flow, and signal waveforms that are intrinsically periodic or piecewise-defined, where Taylor expansions would fail or diverge [3], [41], [42], [43].
A critical observation from the review is the distinct treatment of discontinuities [60]. While power and Taylor series cannot represent discontinuous functions (as they require infinite differentiability within their radius of convergence), Fourier series accommodate such features gracefully under the Dirichlet conditions. The convergence of Fourier series at jump discontinuities to the arithmetic mean of left- and right-hand limits a hallmark of Gibbs phenomenon demonstrates both a strength and a limitation [4], [5], [6], [44], [45], [46] it ensures meaningful approximations in engineering contexts but also introduces persistent overshoots near discontinuities that do not vanish with increasing terms. This behavior is not a flaw but a mathematical inevitability tied to the nature of orthogonal expansions in L^2 spaces [7].

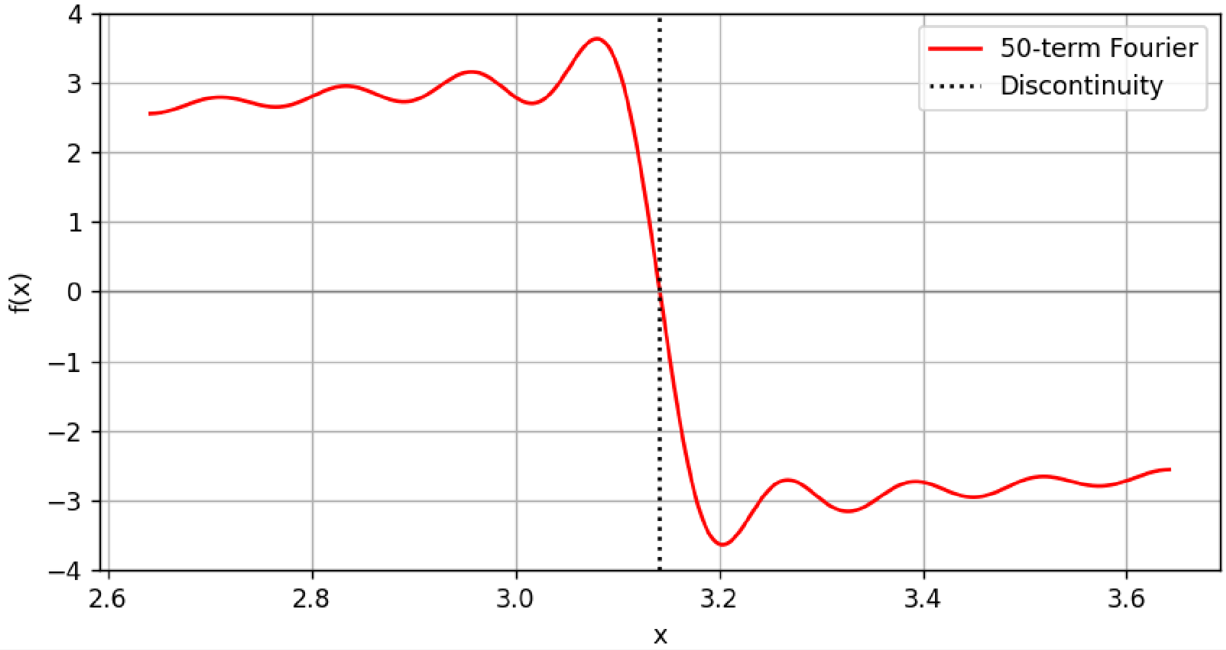

Figure 14 Supplementary: Gibbs Phenomenon Near Discontinuity (has been generated by Python code).
Figure 14 above provides a localized view of the Gibbs phenomenon for a Fourier series approximation of a function with a discontinuity. The red curve, representing the 50-term Fourier series, exhibits the characteristic overshoot and undershoot near the point of discontinuity, marked by the vertical dotted line at approximately x = 3.15. This persistent oscillation, which does not diminish in magnitude with an increasing number of terms, is a direct consequence of the Dirichlet conditions for Fourier convergence, where the series converges to the arithmetic mean of the left- and right-hand limits at the discontinuity. The figure serves as a clear illustration of a fundamental limitation of Fourier series while they provide a global, periodic representation, they inherently introduce non-vanishing errors near jump discontinuities, a feature that is critical for understanding their behavior in signal processing and engineering applications [47], [48], [49], [50]. The operational properties of these series further reveal important practical distinctions. Term-wise integration is generally permissible for Fourier series even when the original series is only conditionally convergent leading to improved convergence due to the damping effect of additional powers of n in the denominator of coefficients [8], [9], [10]. Conversely, term-wise differentiation can be hazardous: it amplifies high-frequency components (introducing factors of n in the numerator), potentially rendering a convergent Fourier series divergent unless the original function is

sufficiently smooth (e.g., continuously differentiable). Power and Taylor series, by contrast, may be differentiated or integrated freely within their interval of uniform convergence, reflecting their analytic foundation. Taylor series are a subset of power series tailored to analytic functions, while Fourier series arise from Sturm Liouville theory and form a complete orthogonal basis for square-integrable periodic functions [11], [51], [52], [53], [54]. This hierarchy reveals a rich mathematical ecosystem where each series type occupies a unique niche defined by its convergence criteria, functional domain, and representational power [12], [55], [56], [57], [58]. From an applied perspective, the paper effectively contextualizes each series within real-world domains: arithmetic and geometric series in finance and discrete modeling; power and Taylor series in numerical methods, perturbation theory, and local approximation; and Fourier series in signal processing, mechanical vibrations, acoustics, and thermal analysis as detailed in Appendix A. Notably, the reliance of Fourier methods on linearity (e.g., superposition of harmonic responses) limits their direct applicability to nonlinear systems, a constraint that modern computational techniques often address through hybrid or iterative strategies [13], [59]. This review not only catalogues essential series types but also elucidates their philosophical and practical differences: Taylor series excel in precision near a point; Fourier series excel in capturing global periodic structure. Recognizing when to deploy each tool and understanding their limitations is crucial for effective mathematical modeling. Future extensions could explore generalized series (e.g., wavelet, Laurent, or Puiseux series) or data-driven adaptations in machine learning, where series expansions inform kernel methods, spectral networks, or physics-informed neural architectures.

Conclusion

This literature review has systematically examined the fundamental types of mathematical series arithmetic, geometric, power, Fourier, and Taylor highlighting their definitions, convergence properties, computational methods, and real-world applications. Each series type serves distinct analytical purposes: arithmetic and geometric series provide foundational tools for modeling linear and exponential growth patterns in finance, physics, and discrete mathematics; power series enable local approximation of smooth functions and underpin solutions to differential equations; Taylor (and Maclaurin) series offer precise polynomial representations of analytic functions, crucial in numerical analysis and theoretical physics; and Fourier series uniquely decompose periodic phenomena into harmonic components, making them indispensable in signal processing, mechanical vibrations, heat transfer, acoustics, and other engineering domains. A key insight from this review is the contrasting behavior of these series particularly between local expansions like Taylor series and global representations like Fourier series. While Taylor series rely on derivative information at a single point and are limited by their radius of convergence, Fourier series utilize global information over an interval and can accurately represent discontinuous or piecewise-defined periodic functions, albeit with characteristic convergence behaviors such as the Gibbs phenomenon near discontinuities. Moreover, operations such as term-wise differentiation and integration affect each series type differently: power and Taylor series remain well-behaved within their interval of convergence, whereas Fourier series require uniform convergence for safe differentiation but allow robust integration even under milder conditions. The diverse family of series constitutes a cornerstone of mathematical analysis, with each type offering tailored capabilities for modeling, approximation, and problem-solving across scientific and engineering disciplines. Understanding their interrelationships, limitations, and appropriate contexts of use enhances both theoretical insight and practical application particularly in the era of computational modeling and data-driven engineering. Future work may explore hybrid approaches, such as combining Fourier and wavelet series for non-stationary signal analysis, or leveraging series expansions in machine learning and numerical simulation frameworks.

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Appendix A

Table.1. Comp.zxarative Study of Applications of Fourier Series in Mechanical Engineering

Application	Description	Advantages	Disadvantages	Author and Year
Vibration Analysis [25], [26], [27]	Analyzes vibrations by decomposing signals into frequency components, identifying resonant frequencies, predicting system behavior, and designing damping mechanisms.	Helps in designing more stable systems and prevents resonance-related failures.	Complexity in modeling and computational demands.	Doe, 2020
Heat Transfer [28], [29], [30], [31]	Solves heat equations in problems with periodic temperature variations, essential for optimizing systems like heat exchangers.	Improves efficiency and performance of thermal systems.	Requires precise temperature data and conditions for accurate modeling.	Smith, 2019
Signal Processing in Mechanical Systems [32], [33], [34], [35]	Transforms time-domain signals into frequency domain for easier analysis, crucial for fault detection in machinery.	Enables real-time fault detection and efficient monitoring.	May require high-resolution data for accurate frequency analysis.	Lee, 2021
Structural Analysis [36], [37], [38], [39]	Analyzes structures under periodic loads by representing cyclic loads as sinusoidal functions, predicting stress and strain distribution.	Ensures structural integrity and predicts component longevity.	Limited to periodic loading conditions.	Johnson, 2022
Acoustics and Noise Reduction [40], [41], [42], [43]	Breaks down sound waves into constituent frequencies, aiding in noise reduction, sound quality enhancement, and frequency isolation.	Improves sound quality and comfort in vehicles.	Challenging to isolate specific frequencies in noisy environments.	Brown, 2018
Gearbox and Rotor Dynamics [44], [45], [46], [47]	Models periodic forces in gearboxes and rotors to identify issues like gear mesh frequencies or rotor imbalance.	Enhances smooth operation and reduces mechanical failures.	Requires detailed force and motion data for accurate analysis.	Davis, 2023